$\pi_1\mathscr{C}$

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January 19, 2026

1 A fibration

Lemma ([Kup], 21.3.1)**.** *There is a fibration (up to weak homotopy)*

$$\mathrm{Diff}_\partial(D^{m+1}) \to \mathscr{C}(D^m) \to \mathrm{Diff}_\partial(D^m)$$

with the first map being the inclusion (diffeomorphisms of $D^m \times I$ that fix the boundary into diffomorphisms of $D^m \times I$ that fix "3 out of 4" sides of the boundary) and the second being restiriction to the upper edge, that is $D^m \times \{1\}$.

If we look at the long exact sequence of homotopy groups we get something that looks like

$$\cdots \to \pi_i\mathscr{C}(D^m) \to \pi_i\mathrm{Diff}_\partial(D^m) \xrightarrow{\partial} \pi_{i-1}\mathrm{Diff}_\partial(D^{m+1}) \to \pi_{i-1}\mathscr{C}(D^m) \to \cdots$$

now we claim that

Lemma. *The connecting homomorphism ∂ is given by λ_i (the Gromoll map).*

If we are interested in disc of origin problems, then we are asking what is the image of this boundary map. But the exactness tells us that this image is the kernel of the map $\pi_{i-1}\mathrm{Diff}_\partial(D^{m+1}) \to \pi_{i-1}\mathscr{C}(D^m)$. Therefore the non-triviality of this kernel "obstructs" the map from being surjective, or whats the same from pulling back a given sphere to having a smaller disc of origin.

Now from [Igu88] we have that some map

$$\mathscr{C}(D^n) \to \mathscr{C}(D^{n+1})$$

is $\max\{(n-4)/3, (n-7)/4\}$ connected. The map into $\pi_0\mathscr{C}(D^n)$ was already shown to be zero (as the previous map has full image), the next obstruction that can occur is the map into $\pi_1\mathscr{C}(D^n)$. To use the stable results we will need then $\max\{(n-4)/3, (n-7)/4\} \geq 1$, which implies that $n \geq 7$.

Lemma. $\pi_1\mathscr{C}(D^n) \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. In our notes on the rational computation of the homotopy groups of diffeomorphisms of discs we have shown that there is a weak equivalence in the stable range between

$$B\mathscr{C}(D^n) \simeq \Omega \mathrm{Wh}^{\mathrm{Diff}}(D^n)$$

in particular

$$\pi_1\mathscr{C}(D^n) = \pi_2 B\mathscr{C}(D^n) = \pi_2\Omega\mathrm{Wh}^{\mathrm{Diff}}(D^n) = \pi_3\mathrm{Wh}^{\mathrm{Diff}}(D^n)$$

by [Kup] the Whitehead spectrum is a weak homotopy invariant of on path connected spaces. Hence $\mathrm{Wh}^{\mathrm{Diff}}(D^n) \simeq \mathrm{Wh}^{\mathrm{Diff}}(*)$. We now proceed by computing the local peices of the Whitehead spaces homotopy groups.

[Wal82, Remark under Cor 3.4] gives us that for all primes p and all $j < 2p-3$ there is an isomorphism between the localised groups

$$\pi_j\mathrm{Wh}^{\mathrm{Diff}}(*)_{(p)} \cong K_j(\mathbb{Z})_{(p)}$$

Because we are interested in $j = 3$ we get that for all p prime and $p \geq 5$ this isomorphism holds. Now from the known values of the $K(\mathbb{Z})$ we have that $K_3(Z) = \mathbb{Z}/48\mathbb{Z} = \mathbb{Z}/(2^4 \times 3)\mathbb{Z}$. Hence $\pi_3\mathrm{Wh}^{\mathrm{Diff}}(*)_{(p)} = 0$ for primes ≥ 5. Thus it remains to compute the local peices for the primes 2 and 3.

[Rog02, Thm 5.8] computes that

$$\pi_3\mathrm{Wh}^{\mathrm{Diff}}(*)_{(2)} \cong \mathbb{Z}/2\mathbb{Z}$$

and [Rog03, Cor 4.9.(b)] computes that

$$\pi_3\mathrm{Wh}^{\mathrm{Diff}}(*)_{(3)} \cong 0$$

Note that this result in Rognes was stated as conditional on the Quillen–Lichtenbaum conjecture, however this was soon after proven by Voevodsky, and therefore this result now holds unconditionally.

□

Remark. [Hat78] claims that $\pi_1\mathscr{C}(D^n)$ is either $(\mathbb{Z}/2\mathbb{Z})^2$ or $\mathbb{Z}/4\mathbb{Z}$, for n large using the results from an from an unpublished paper of his student Igusa. **These results are wrong.** They are apparently corrected in [Jah10].

2 Higher Homotopy Groups

We always are in the stable range. So we use $\pi_i\mathscr{C}(D^n) \cong \pi_{i+2}\mathrm{Wh}^{\mathrm{Diff}}(*)$.

2.1 π_2, π_4

We need π_4 and π_6 of $\mathrm{Wh}^{\mathrm{Diff}}(*)$, using the same idea as above we have that it is isomorphic at all primes ≥ 5 to $K_4(\mathbb{Z}) = K_6(\mathbb{Z}) = 0$. Hence we just need the 2 and 3 primary parts of π_4 and π_6 of $\mathrm{Wh}^{\mathrm{Diff}}(*)$, which Rognes kindly tells us are:

$$\pi_4\mathrm{Wh}^{\mathrm{Diff}}(*)_{(2)} = 0, \quad \pi_4\mathrm{Wh}^{\mathrm{Diff}}(*)_{(3)} = 0$$

$$\pi_6\mathrm{Wh}^{\mathrm{Diff}}(*)_{(2)} = 0, \quad \pi_6\mathrm{Wh}^{\mathrm{Diff}}(*)_{(3)} = 0$$

Hence stably $\pi_2\mathscr{C}(D^n) \cong \pi_4\mathscr{C}(D^n) = 0$.

2.2 π_3

Again all primes ≥ 5 are given by the $K_5(\mathbb{Z}) = \mathbb{Z}$. So just need the two and three primary parts given by

$$\pi_5 \mathrm{Wh}^{\mathrm{Diff}}(*)_{(2)} = \mathbb{Z}, \quad \pi_5 \mathrm{Wh}^{\mathrm{Diff}}(*)_{(3)} = 0$$

So we conclude that $\pi_3 \mathscr{C}(D^n) \cong \pi_5 \mathrm{Wh}^{\mathrm{Diff}}(*) \cong K_5(\mathbb{Z}) = \mathbb{Z}$ (just to be clear they are all rationally $\mathbb{Q}$ and the torsion parts vanish so they must be $\mathbb{Z}$, given they are finitely generated abelian groups).

2.3 π_5

In this case we are dealing with $K_7(\mathbb{Z}) = \mathbb{Z}_{240} = \mathbb{Z}_{2^4 \times 3 \times 5} = \mathbb{Z}_{2^4} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$. In this case the inequality is for $p > 5$ and hence all torsion for those primes vanish, as they dont exist in $K_7(\mathbb{Z})$. Finally Rognes tells us that the first 5 torsion is in degree $\pi_{18} \mathrm{Wh}^{\mathrm{Diff}}(*)$ the first 3 torsion is in degree 11. Thus it is only two torsion again given by $\mathbb{Z}_2$.

Remark. This method, using Rognes results on odd regular primes, will get you the first 68 homotopy groups of the concordance space, provided you are not 2 mod 4. At this point we will get that $69 + 2 \not< 2(37) - 3$ where 37 is the first irregular prime. Thus we cannot use our comparison to K theory to show that the torsion at this prime vanishes. Perhaps there are other methods.

3 The Concordance Obstruction

The idea then is to look at the LES for the fibration above and consider the group $\pi_1 \mathscr{C}(D^n)$ as an obstruction to pulling back in the Gromoll filtration. Because the group $\pi_i \mathrm{Diff}_\partial(D^n)$ appears in the LES above in two places with shifting indicies we can sort of weave them together:

$$\begin{tikzcd}
 & \vdots \arrow[d] & & & \vdots \arrow[d] & & \\
 & \pi_1 \mathrm{Diff}_\partial(D^{n+2}) \arrow[d] & & & \pi_0 \mathrm{Diff}_\partial(D^{n+2}) \arrow[d] & & \\
 & \pi_1 \mathscr{C}(D^{n+1}) \arrow[d, "r"] \arrow[dr, dashed, "or"] & & & \pi_0 \mathscr{C}(D^{n+1}) \arrow[d] & & \\
\cdots \arrow[r] & \pi_1 \mathrm{Diff}_\partial(D^{n+1}) \arrow[r, "o"] & \pi_1 \mathscr{C}(D^n) \arrow[r] & \pi_1 \mathrm{Diff}_\partial(D^n) \arrow[r] & \pi_0 \mathrm{Diff}_\partial(D^{n+1}) \arrow[r] & \pi_0 \mathscr{C}(D^n) \arrow[r] & \pi_0 \mathrm{Diff}_\partial(D^n)
\end{tikzcd}$$

where the verticle sequences continue beneath the horrozontal as well. This is looking something like a spectral sequence and indeed this is the idea behind [Hat78] spectral sequence. The map *or* is the differential map in the exact couple he constructs. Now we can fill in some of these groups explicitly

$$\begin{tikzcd}
 & & \vdots \arrow[d] & & & & \\
 & & \pi_1 \mathrm{Diff}_\partial(D^{n+2}) \arrow[d] & & & & \\
 & & \mathbb{Z}_2 \arrow[d, "r"] \arrow[dr, dashed, "or"] & & & 0 \arrow[d] & \\
\cdots \arrow[r] & \pi_2 \mathrm{Diff}_\partial(D^n) \arrow[r] & \pi_1 \mathrm{Diff}_\partial(D^{n+1}) \arrow[r, "o"] \arrow[d, two heads] & \mathbb{Z}_2 \arrow[r] & \pi_1 \mathrm{Diff}_\partial(D^n) \arrow[r, two heads] & \pi_0 \mathrm{Diff}_\partial(D^{n+1}) \arrow[r] & 0 \\
 & & \pi_0 \mathrm{Diff}_\partial(D^{n+2}) & & & &
\end{tikzcd}$$

and our goal is to study this first non-trivial crossing.

References

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